WEAK TIGHT GEODESICS IN THE CURVE COMPLEX

by

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ABSTRACT

Let $S_{g,n}$ be a compact surface of g genus and n boundary components. Let $\xi(S_{g,n}) = 3g + n - 3$ be the complexity of the surface. Our main space in this dissertation is the curve complex $C(S)$ by Harvey. The curve complex is known to be a Gromov hyperbolic, infinite diameter, and locally infinite space. Our main object in this dissertation is tight geodesics by Masur-Minsky. The curve complex plays an important role in the study of low dimensional topology and geometry. Especially, the Masur-Minsky theory of hierarchies of the curve complex gave a complete understanding on the large-scale geometry of the mapping class groups with other important tools such as tight geodesics and subsurface projections. Bowditch studied the cardinality of slices of tight geodesics and as its applications, he showed that the mapping class groups act on the curve complex acylindrically and that the stable lengths of pseudo-anosov elements are rational with bounded denominator. However, since his proof is done by a geometric limit argument via hyperbolic 3-manifolds, his argument does not give a computable bound for the cardinality of slices of tight geodesics. In this dissertation, we extend Bowditch's result. In Chapter 2, we show there exists a computable bound of slices of tight geodesics which only depends on the surface and the distance between initial and terminal curves by a combinatorial approach. In Chapter 3, we show there exists a computable bound of slices of tight geodesics which only depends on the surface. Indeed, the second statement is a direct corollary of the following new result. While the curve complex is locally infinite, we show that it is "locally finite" under subsurface projections. We define the local finiteness property; this is a property which any locally finite graph whose diameter is infinite with a uniformly bounded valency satisfies. Suppose X is a such graph. Let d_X be the simplicial metric on X. Local finiteness property: Given $l > 0$ and $k > 1$, there exists a computable $N(l, k, \text{valency of } X) > 0$ such that for any $C \subseteq X$, if $|C| > N$, then there exist $\{x_j\} \subset C$ such that $|\{x_j\}| \geq k$ so that $d_X(x_s, x_t) > l$ for all s, t such

that $s \neq t$. The local finiteness property does not hold for the curve complex since it is locally infinite. However, by using subsurface projections, we show Local finiteness property of the curve complex via subsurface projections: Given $l > 0$ and $k > 1$, there exists a computable $N(l, k, \xi(S)) > 0$ such that for any $C \subseteq C(S)$, if $|C| > N$, then there exists $\{x_j\} \subseteq C$ such that $|\{x_j\}| \geq k$ and $Z \subseteq S$ so that $d_Z(x_s, x_t) > l$ for all s, t such that $s \neq t$. As a corollary of the above main result with a special behavior of tight geodesics, we give a computable bound of slices of tight geodesics which only depends on the surface. Lastly, we define a new class of geodesics, weak tight geodesics. Indeed, we can use the local finiteness property of the curve complex via subsurface projections to show that the cardinality of slices of weak tight geodesics are bounded by $W^{(W^{\xi(S)})}$ for some constant W which only depends on the surface.

To my parents

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CHAPTER 1

INTRODUCTION

Let $S = S_{g,n}$ be a surface with g genus and n boundary components, $\xi(S_{g,n}) = 3g + n - 3$ be the complexity, and $\chi(S_{g,n})$ be the Euler characteristic of $S_{g,n}$. We assume all curves are simple, closed, essential, and nonperipheral. Also we assume all arcs are simple, essential, and nonperipheral and the isotopy of arcs is relative to the boundaries setwise unless we say relative to the boundaries pointwise.

Harvey associated the set of curves in a surface with a simplicial complex, the curve complex [Har81]. Suppose $\xi(S) \geq 1$. The vertices are isotopy classes of curves and the simplices are collections of curves that can be mutually realized to intersect the minimal possible geometric intersection number, 0 for $\xi(S) > 1$, 1 for $S = S_{1,1}$ and 2 for $S = S_{0,4}$. We also review the arc complex, $A(S)$ and the arc and curve complex, $AC(S)$. Suppose $\xi(S) \geq 0$, the vertices of $A(S)$ ($AC(S)$) are isotopy classes of arcs (arcs and curves) and the simplices are collections of arcs (arcs and curves) that can be mutually realized to be disjoint in S. In this dissertation, $a \in C(S)$, $a \in A(S)$, and $a \in AC(S)$ means that a is an element of 0-skelton in each complex.

Indeed, every complex above is a geodesic metric space with the simplicial metric. If $x, y \in C(S)$, we let $d_S(x, y)$ be the length of a geodesic between x and y. If $A, B \subset C(S)$, we define

$$d_S(A, B) = \max_{a \in A, b \in B} d_S(a, b).$$

Suppose $x, y \in AC(S)$, the intersection number, $i(x, y)$ is the minimal geometric intersection number under the isotopy classes of x and y, we say x and y are in minimal position if they realize the intersection number.

Definition 1.1. Let X be a geodesic metric space. We say X is a Gromov-hyperbolic space with the hyperbolicity constant δ (or in short, we say δ-hyperbolic space) if

$\triangle_{a,b,c}$ is a triangle whose vertices are $a, b, c \in X$, then any edge of $\triangle_{a,b,c}$ is contained in the δ-neighborhood of the other two edges.

Indeed, the curve complex above had been known to be Gromov-hyperbolic [MM99] [Bow06] where the hyperbolicity constant grows with $\xi(S)$, but recently the constant has been shown to be uniform for all surfaces [Aou13], [Bow14], [MCS], [HPW].

Now, we review our main object in this dissertation, tight geodesics by Masur-Minsky.

1.1 Tight geodesics

A multicurve is the set of curves that form a simplex in the curve complex.

Definition 1.2. Suppose V and W are multicurves in S.

- Let $A \subseteq S$, then $R(A)$ is the regular neighborhood of A in S.
- We say V and W fill S if there is no curve in the complementary components of $R(V \cup W)$ in S.
- A unique subsurface, $F(V, W)$, which V and W fill is constructed by taking $R(V \cup W)$ and filling in a disk for every complementary component of $R(V \cup W)$ in S which is a disk. We note that V and W fill S if and only of $F(V, W) = S$ up to isotopy.

We observe

Lemma 1.3. *Suppose $\xi(S) > 1$. Let V and W be multicurves in S. Then V and W fill S if and only if $d_S(V, W) > 2$.*

Proof. V and W fill S if and only if any curve which is not contained in $V \cup W \subseteq C(S)$ intersects some curve in V or W. $\square$

Now, we define tight geodesics.

Definition 1.4.

- A multigeodesic is a sequence of multicurves $\{V_i\}$ such that $d_S(a, b) = |p - q|$ for all $a \in V_p$ and $b \in V_q$ for all p, q such that $p \neq q$.

- A tight multigeodesic is a multigeodesic $\{V_i\}$ such that $V_i = \partial F(V_{i-1}, V_{i+1})$ for all i. Let $x, y \in C(S)$. A tight geodesic between x and y is a geodesic $\{x_i\}$ such that $x_i \in V_i$ for all i where $\{V_i\}$ is some tight multigeodesic between x and y.

Remark 1.5. We remark that some authors refer to tight multigeodesics as tight geodesics while by our definition, tight geodesics are indeed geodesics in the curve complex. Essentially, these two perspectives are equivalent so we will not distinguish them in this dissertation.

1.2 Some results on tight geodesics prior to this dissertation

We recall the results regarding the cardinality of slices of tight geodesics prior to this dissertation; these are due to Masur-Minsky, Bowditch, Schackleton, and Webb in chronological order. We also state some applications of these studies.

Masur-Minsky proved that there exists at least one and only finitely many tight geodesics between any pair of curves [MM00].

Bowditch defined a slice of the union of tight geodesics between a pair of curves and a pair of the set of curves and showed that there exists a uniform but non-computable bound for the slice [Bow08]; we remark his bound was not computable because he uses a geometric limit argument using 3-dimensional hyperbolic geometry. We also refer to [Bow07] which involves a geometric limit argument.

Schackleton showed that there exists a computable bound for the slice which depends on the intersection number of a given pair of curves [Sha12]. The idea of his proof extends to show that there exists an algorithm to compute the distance between a given pair of curves. For this result, we also refer to the unpublished dissertation of Leasure [Lea02].

Independently, Webb [Weba] and the author [Watb] showed that there exists a computable bound for the slice which depends on the distance between a given pair of curves, and it is one of the key points in [Weba], where he showed that there exists a uniform and computable bound for the slices by a combinatorial argument. We give an another proof to obtain a uniform and computable bound for the slices.

The main result of this dissertation is to overcome the fact that the curve complex is locally infinite; we call this local finiteness property of the curve complex via subsurface projections. Obtaining a uniform and computable bound for the slices is a direct corollary of this property.

1.3 Weak tight geodesics

In this dissertation, we define a new class of geodesics, weak tight geodesics. Let $x, y \in C(S)$. We say a geodesic between x and y is a D-weakly tight geodesic if for any vertex a on the geodesic, if $\pi_Z(a) \neq \emptyset$ for $Z \subsetneq S$, then there exists a uniform $D > 0$ such that $d_Z(x, a) \leq D$ or $d_Z(a, y) \leq D$. Here, uniform is in the sense of choice of vertices and subsurfaces. We will observe that tight geodesics are M-weakly tight geodesics. We also note that every geodesic is a D-weakly tight geodesic for some D. In this dissertation, we will observe that there exists a uniform and computable bound for the slices of D-weakly tight geodesic for any given two curves by local finiteness property of the curve complex via subsurface projections.

These studies of tight geodesics have many applications, including Thurston's ending lamination conjecture [BCM12] [Min10], the asymptotic dimension of the curve graph [BF08], and the stable lengths of pseudo-Anosov elements [Bow08]. The author used tight geodesics to study intersection numbers between two curves via subsurface projections [Wata].

CHAPTER 2

A POLYGON DECOMPOSITION BY FILLING CURVES AND TIGHT GEODESICS

Let $C^{\triangle}(S)$ be the set of all simplicies in the curve complex. We define slices.

Definition 2.1. Suppose $\alpha, \beta \in C(S)$. By choosing the starting point, $X_0^T = \alpha$, X_i^T is the set of elements in $C^{\triangle}(S)$ which are on tight geodesics between α and β and distance i apart from $X_0^T = \alpha$. We call X_i^T a slice.

We remark that the slices which we use in the next chapter (Bowditch's slice) will be different from the above. However, they can be understood as two equivalent definitions. In the next chapter, we use balls as our slices. We note that we can use balls as our slices since the curve complex is a Gromov hyperbolic space. We discuss this more in the next chapter.

It is straightforward to see that if $\alpha, \beta \in C(S)$ such that $d_S(\alpha, \beta) = 2$, then $|X_1^T| = 1$. Therefore, we consider the case when two curves are distance more than 2 apart, i.e., when they fill a surface. We state the main theorem in this chapter.

Theorem 2.2. *Suppose $\alpha, \beta \in C(S)$ such that $d_S(\alpha, \beta) > 2$. By choosing $X_0^T = \alpha$, there exists a bound on $|X_i^T|$ which depends on $d_S(\alpha, \beta)$ and the topology of S for all i.*

Lastly, we remark that the above theorem was proved independently by Webb [Weba].

2.1 The number of arcs

We will develop the machinery for the computation for our main result. Suppose $V \in C^{\triangle}(S)$. We say an arc γ is properly embedded relative to V if γ does not bound a disk in S with any subinterval of V. We denote $A(S, V)$ as the the set of homotopy classes of properly embedded arcs relative to V. Furthermore, we denote $A^{\triangle}(S,V)$ as the set of the collections of mutually disjoint elements in $A(S,V)$. If a simplex $\sigma \in A^{\triangle}(S,V)$ is such that $S-(\sigma \cup V)$ is union of disks and punctured disks, we call such a simplex a filling simplex.

The following lemma is the key in this chapter.

Lemma 2.3. *Suppose $V \in C^{\triangle}(S)$. Let $\sigma \in A^{\triangle}(S,V)$ be a filling simplex. Then there exists K_S such that*

$$|\{\sigma' \in A^{\triangle}(S,V)|i(\sigma,\sigma') = \max_{v\in\sigma, v'\in\sigma'} i(v,v') = 0\}| \leq K_S.$$

Proof. By definition of σ, $S-(\sigma\cup V)$ is the union of disks and punctured disks, which can be understood by polygons whose edges consists of the elements of σ and the compact subintervals of elements in V. We observe that each polygon has an even number of edges possibly with a punctured in its interior. Furthermore, we observe that there is no 2-gon without a puncture since all the elements in σ are properly embedded relative to V. Also, there is no 4-gon without a puncture, otherwise two of its edges from σ would be homotopic relative to V, but by the definition of $A^{\triangle}(S,V)$, the elements in σ are homotopically distinct. Therefore, $S-(\sigma\cup V)$ is the union of $2m$-gons for $m\geq 1$ with a puncture and $2m$-gons for $m\geq 3$ without a puncture.

First, we count the number of arcs in each polygon. Since every element in σ is an edge of polygons, those arcs miss σ. In a punctured $2m$-gon, we count at most $2\binom{m}{2}$ arcs for the arcs whose boundaries lie on the different edges from V in the polygon, and m arcs for the arcs whose boundaries lie on the same edge from V in the polygon. In a $2m$-gons without a puncture, we count at most $\binom{m}{2}$ arcs.

Lastly, we observe that there are finitely many types of polygon decompositions as the compliment of σ and V can occurs by simple Euler characteristic argument. In particular, we observe that a $2m$-gon without a puncture contributes to $\chi(S)$ by

$$\frac{|vertices|}{4} - \frac{|edges|}{2} + |faces| = \frac{-m}{2} + 1$$

and a $2m$-gon with a puncture contributes to $\chi(S)$ by $\frac{-m}{2}$. Therefore, it is straightforward to see that there exists K_S such that

$$|\{\sigma' \in A^{\triangle}(S,V)|i(\sigma,\sigma') = 0\}| \leq K_S,$$

and we are done.

□

Now, we will use Lemma 2.3 to compute a bound of the number of tight geodesics.

Definition 2.4. Suppose $V \in C^{\triangle}(S)$. We let

$$C(S,V) = \{x \in C(S) | i(x,V) = \max_{v \in V} i(x,v) > 0\}$$

and $C^{\triangle}(S,V)$ be the set of simplicies in $C(S,V)$.

We define $I_V : C^{\triangle}(S,V) \to A^{\triangle}(S,V)$. Let $W \in C^{\triangle}(S,V)$. Then we define

$$I_V(W) = \{W \cap (S - V)\} / \sim$$

where $r_1 \sim r_2$ if they are homotopic relative to V.

Now, we observe the following lemma with Lemma 2.3.

Lemma 2.5. *Let $W, V \in C^{\triangle}(S)$ be such that there exists a tight geodesic of distance k for $k \geq 3$ between them. We let $W = X_0^T$ and $I_V(X_i^T) = \cup_{U \in X_i^T} I_V(U)$. Then,*

$$I_V(X_{i+1}^T) \subseteq \cup_{\sigma \in I_V(X_i^T)} \{\sigma' \in A^{\triangle}(S,V) | i(\sigma,\sigma') = 0\} \text{ for all } i \leq k-3.$$

Furthermore,

$$|I_V(X_{i+1}^T)| \leq |I_V(X_i^T)| \cdot K_S \text{ for all } i \leq k-3.$$

Proof. Suppose $i \leq k-3$. Let $U \in X_{i+1}^T$. Then there exists $U' \in X_i^T$ that lie on the same tight geodesic, so we have $i(I_V(U), I_V(U')) = 0$.

Also since $d_S(U',V) \geq 3$, $I_V(U')$ is a filling simplex, therefore the second statement directly follows by Lemma 2.3. □

By the induction with Lemma 2.5, we have

Corollary 2.6. *Let $W, V \in C^{\triangle}(S)$ be such that there exist a tight geodesic of distance k for $k \geq 3$ between them. We let $W = X_0^T$, then $|I_V(X_i^T)| \leq K_S^i$, for all $i \leq k-2$.*

In particular, we have

Theorem 2.7. *Let $x, y \in C(S)$ be such that $d_s(x,y) = k$ for $k \geq 3$.We let $\alpha = X_0^T$, then $|I_\beta(X_{k-2}^T)| \leq K_S^{k-2}$.*

2.2 Computation of $|X_1^T|$, $|X_{k-1}^T|$ and $|X_i^T|$

Suppose $V \in C^{\triangle}(S)$ and A is a set of properly embedded arcs relative to V. We similarly define a unique subsurface that A and V fills, $S(A,V)$, by first taking the regular neighborhood of $A \cup V$, and filling in disks or annuli on each component of the complement of the regular neighborhood in S depending on whether a complement is a disk or not.

First, we compute $|X_{k-1}^T|$.

Theorem 2.8. *Let $W, V \in C^{\triangle}(S)$ be such that there exist a mutigeodesic of distance 2 between them. Then $S(V,W)$ and $S(V, I_V(W))$ are homeomorphic in S. In particular,*

$$\partial(S(V,W)) = \partial\Big(S(V, I_V(W))\Big) \text{ in } C(S).$$

Proof. Let $A_W = \{W \cap (S - V)\}$. Then we observe $S(V,W) = S(V, A_W)$. If $A_W = I_V(W)$, we are done. If not, there exists $A'_W \subset A(S,V)$ such that

$$I_V(W) \subseteq A'_W \subsetneq A_W$$

and

$$|A_W| = |A'_W| + 1.$$

We observe $S(V, A_W)$ and $S(V, A'_W)$ are homeomorphic in S and we are done.

□

By the above observation, we have

Theorem 2.9. *Let $x, y \in C(S)$ be such that $d_s(x,y) = k$ for $k \geq 3$. We let $\alpha = X_0^T$, then $|X_{k-1}^T| \leq K_S^{k-2}$.*

Proof. By the definition of a tight geodesic, $X_{k-1}^T = \cup_{W \in X_{k-2}^T} \partial(S(\beta, W))$. By Theorem 2.8, we have

$$\cup_{W \in X_{k-2}^T} \partial(S(\beta, W)) = \cup_{W \in X_{k-2}^T} \partial\Big(S(\beta, I_\beta(W))\Big),$$

but $|I_\beta(X_{k-2}^T)| \leq K_S^{k-2}$ by Theorem 2.7. □

We observe that the symmetric argument gives the same bound for $|X_1^T|$, which leads to the $|X_t^T|$ for general t.

Theorem 2.10. *Let $x, y \in C(S)$ be such that $d_s(x, y) = k$ for $k \geq 3$.We let $\alpha = X_0^T$ and let $t \leq \lfloor \frac{k}{2} \rfloor$, then $|X_t^T|$ and $|X_{k-t}^T|$ are both bounded by $\prod_{i=1}^{t} K_S^{k-(1+i)}$.*

CHAPTER 3

THE LOCAL FINITENESS THEOREM OF THE CURVE GRAPH AND WEAK TIGHT GEODESICS

First, we review subsurface projections from [MM00]. It will be a main tool to understand tight geodesics so that we can relate the cardinality of slices of tight geodesics with the local finiteness property of the curve graph. The goal of this chapter is to obtain a uniform bound for the slices of weak tight geodesics.

3.1 Subsurface projections

Suppose A is a subset of S. We let R(A) be a regular neighborhood of A in S. Let $\mathcal{P}(C(S))$ and $\mathcal{P}(AC(S))$ be the set of finite subsets in each complex. We define subsurface projections. Let Z be a subsurface of S such that $\xi(Z) \neq 0$ and $\xi(Z) \geq -1$.

3.1.1 Nonannular projections

Suppose $\xi(Z) > 0$. Let $x \in AC(S)$ and assume x and ∂Z are in minimal position. We define the map

$$i_Z : AC(S) \to \mathcal{P}(AC(Z))$$

such that if $x \in AC(S)$, then $i_Z(x)$ is the set of arcs and curves obtained by $x \cap Z$. Also we define the map

$$p_Z : AC(Z) \to \mathcal{P}(C(Z))$$

such that if $x \in AC(Z)$, then $p_Z(x) = \partial R(x \cup z \cup z')$ where z, z' are components of $\partial(Z)$ such that $z \cap \partial(x) \neq \emptyset$ and $z' \cap \partial(x) \neq \emptyset$. We note z could be the same as z'.

Also if $x \in C(Z)$, then $p_Z(x) = \partial R(x)$, which is just x. We observe $|\{p_Z(x)\}| \leq 2$. If $C \subset AC(Z)$, we define

$$p_Z(C) = \bigcup_{c \in C} p_Z(c).$$

The subsurface projection to Z is the map

$$\pi_Z = p_Z \circ i_Z : AC(S) \to \mathcal{P}(C(Z)).$$

If $C \subset AC(S)$, we define

$$\pi_Z(C) = \bigcup_{c \in C} \pi_Z(c).$$

3.1.2 Annular projections

Suppose Z is an essential annulus in S. Fix a hyperbolic metric on S, compactify the cover of S which corresponds to $\pi_1(Z)$ with its Gromov boundary, and denote the resulting surface S^Z. We define the vertices of $C(Z)$ to be the set of the isotopy classes of arcs whose endpoints lie on two boundaries of S^Z; here, the isotopy is relative to ∂S^Z pointwise. Two vertices of $C(Z)$ are distance 1 apart if they can be isotoped to be disjoint in the interior of S^Z.

The subsurface projection to Z is the map

$$\pi_Z : AC(S) \to \mathcal{P}(C(Z))$$

such that if $x \in AC(S)$, then $\pi_Z(x)$ is the set of all arcs obtained by the lift of x which connects two boundaries of S^Z. As in the previous case, if $C \subset AC(S)$, we let

$$\pi_Z(C) = \bigcup_{c \in C} \pi_Z(c).$$

3.1.3 Subsurface projection distances

We recall important results regarding subsurface projection distances. First, we define subsurface projection distances. If $A, B \subset AC(S)$, we let

$$d_Z(A, B) = \max_{a \in \pi_Z(A), b \in \pi_Z(B)} d_Z(a, b).$$

In this paper, if $Z \subset S$ and $x \in AC(S)$ such that $\pi_Z(x) = \emptyset$, then we say x misses Z or Z misses x.

Now, we observe the following lemma for annular projections. First, we recall the following.

Definition 3.1. The mapping class group of a surface S, $Mod(S)$, is the group of all orientation preserving self-homeomorphisms of S up to isotopy.

Suppose Z is an essential annulus in S and let the core curve of Z be $x \in C(S)$. We topologically understand Z by $\{x\} \times [0,1] = S^1 \times [0,1]$. Then the Dehn twist along x, T_x is an element in $Mod(S)$ where

$$T_x(a) = \begin{cases} a & \text{if } a \notin Z \\ (e^{2i\pi(\theta+r)}, r) & \text{if } a = (e^{2i\pi(\theta)}, r) \in Z = S^1 \times [0,1] \end{cases}$$

The following lemma states a relation between Dehn twists and annular projections. We recall

Lemma 3.2 ([MM00]). *Suppose Z is an essential annulus in S and the core curve of Z is $x \in C(S)$. Let T_x be the Dehn twist along x. If $y \in C(S)$ such that $\pi_Z(y) \neq \emptyset$, then*

$$d_Z(y, T_x^n(y)) = |n| + 2 \textit{ for } n \neq 0.$$

If y intersects x exactly twice with opposite orientation, a half twist along x to y is well defined to obtain a curve $H_x(y)$, which is taking $x \cup y$ and resolving the intersections in a way consistent with the orientation (see [Luo10] for a generalization). Then $H_x^2(y) = T_x(y)$, and

$$d_Z(y, H_x^n(y)) = \left\lfloor \frac{|n|}{2} \right\rfloor + 2 \textit{ for } n \neq 0.$$

Lastly, we observe the Bounded Geodesic Image Theorem which was first proved by Masur-Minsky [MM00] and recently by Webb [Webb] by a more direct approach.

Theorem 3.3. *(Bounded Geodesic Image Theorem) Let δ be a hyperbolicity constant of the curve graph of S. There exists $M(\delta)$ such that if $\{x_i\}_0^n$ is a geodesic and $\pi_Z(x_i) \neq \emptyset$ for $Z \subset S$ for all $0 \leq i \leq n$, then*

$$d_Z(x_0, x_n) \leq M.$$

In the rest of this paper, we mean M as M in the statement of the Bounded Geodesic Image Theorem. Here, we note that M is uniform for all surfaces since it only depends on a hyperbolicity constant which is uniform.

We observe the following lemma; it states a special behavior of tight geodesics under subsurface projections.

Lemma 3.4. *Suppose $x, y \in C(S)$. Let $\{V_j\}$ be a tight multigeodesic between x and y. If $\pi_Z(V_i) \neq \emptyset$ for some $Z \subsetneq S$, then*

$$d_Z(x, V_i) \leq M \text{ or } d_Z(V_i, y) \leq M.$$

In particular, the above statement holds if $\{V_j\}$ is a tight geodesic.

Proof. The proof of the second statement is a restatement of that of the first statement, so we only prove the first statement.

We assume $\xi(S) > 1$. The proof for $\xi(S) = 1$ directly follows from the argument given in this proof.

Suppose $\pi_Z(V_j) \neq \emptyset$ for all $j > i$. Then by the Bounded Geodesic Image Theorem, we have

$$d_Z(V_i, y) \leq M.$$

Suppose $\pi_Z(V_k) = \emptyset$ for some $k > i$. We have two cases.

<u>If $k > i + 1$:</u> By Lemma 1.3, we observe $\pi_Z(V_j) \neq \emptyset$ for all $j < i$ since $d_S(V_k, V_j) > 2$, i.e., V_k and V_j fill S. Now, by the Bounded Geodesic Image Theorem, we have

$$d_Z(x, V_i) \leq M.$$

<u>If $k = i + 1$:</u> By tightness, we have $V_i = \partial F(V_{i-1}, V_{i+1})$. Since $\pi_Z(V_i) \neq \emptyset$ and $\pi_Z(V_{i+1}) = \emptyset$, we must have $\pi_Z(V_{i-1}) \neq \emptyset$. Now, we repeat the argument by using Lemma 1.3 and the Bounded Geodesic Image Theorem, and we have

$$d_Z(x, V_i) \leq M.$$

□

3.2 Weak tight geodesics

We define a new class of geodesics, weak tight geodesics. Let $x, y \in C(S)$. We say a geodesic between x and y is a D-weakly tight geodesic if for any vertex a on the geodesic, if $\pi_Z(a) \neq \emptyset$ for $Z \subsetneq S$, then there exists a uniform $D > 0$ such that $d_Z(x, a) \leq D$ or $d_Z(a, y) \leq D$. Here, uniform is in the sense of choice of vertices and subsurfaces.

By Lemma 3.4, we have

Proposition 3.5. *Tight geodesics are M-weakly tight geodesics.*

We also note every geodesic is a D-weakly tight geodesic for some D.

In this chapter, we will observe that there exists a uniform and computable bound for the slices of D-weakly tight geodesics for any given two curves by local finiteness property of the curve complex via subsurface projections.

3.3 Bowditch's slices and its uniform bound for weak tight geodesics

We review the definition of Bowditch's slices from [Bow08] and state our result. Let $N_i(x)$ denote the i-ball around $x \in C(S)$. Suppose $a, b \in C(S)$. Let $\mathcal{L}_T(a, b)$ be the set of all tight geodesics between a and b, and

$$G(a, b) = \bigcup \mathcal{L}_T(a, b) \subseteq C(S).$$

Suppose $A, B \subseteq C(S)$. Let

$$\mathcal{L}_T(A, B) = \bigcup_{a \in A, b \in B} \mathcal{L}_T(a, b)$$

and

$$G(A, B) = \bigcup_{a \in A, b \in B} G(a, b) \subseteq C(S).$$

Suppose $a, b \in C(S)$ and $r > 0$. Let

$$G(a, b; r) = G(N_r(a), N_r(b)).$$

The following result is due to Bowditch [Bow08] without computable bounds. Here, we state the recent result by Webb.

Suppose $a, b \in C(S)$. We let $g_{a,b}$ be a geodesic between a and b.

Theorem 3.6 ([Weba]). *Suppose $\xi(S) > 1$.*

1. *For any $a, b \in C(S)$ and any $c \in C(S)$ such that c lies on some $g_{a,b}$,*

$$|G(a,b) \cap N_\delta(c)| \leq K^{\xi(S)}$$

where K is a uniform constant.

2. *For any $r \geq 0$, $a, b \in C(S)$ such that $d_S(a,b) \geq 2r + 2j + 1$ (where $j = 10\delta + 1$), then for any $c \in C(S)$ such that c lies on some $g_{a,b}$ and $c \notin N_{r+j}(a) \cup N_{r+j}(b)$,*

$$|G(a,b;r) \cap N_{2\delta}(c)| \leq K^{\xi(S)}$$

where K is a uniform constant.

Remark 3.7. We observe that $N_\delta(c)$ intersects all (tight) geodesics between a and b, $N_{2\delta}(c)$ intersects all (tight) geodesics between $N_r(a)$ and $N_r(b)$ by the hyperbolicity of the curve graph, so we call $G(a,b) \cap N_\delta(c)$ and $G(a,b;r) \cap N_{2\delta}(c)$ slices. We note that these Bowditch's slices are essentially the same as the slices defined in the previous chapter.

In this chapter, we show that there exist a uniform and computable bound for the slices of weak tight geodesics. It is a direct corollary of the local finiteness property of the curve complex via subsurface projections. (In this chapter, we prove a contrapositive version of the local finiteness property, which we rephrase by Theorem A.) Also in our statement, the hypothesis of the second statement will be weaker, i.e., j will be $3\delta + 2$ instead of $10\delta + 1$.

We define similar notations for slices of D-weak tight geodesics. Suppose $a, b \in C(S)$. Let $\mathcal{L}^D_{WT}(a,b)$ be the set of all D-weak tight geodesics between a and b, and

$$G^D(a,b) = \bigcup \mathcal{L}^D_{WT}(a,b) \subseteq C(S).$$

Also suppose $A, B \subseteq C(S)$. Let

$$\mathcal{L}^D_{WT}(A,B) = \bigcup_{a \in A, b \in B} \mathcal{L}^D_{WT}(a,b)$$

and

$$G^D(A,B) = \bigcup_{a \in A, b \in B} G^D(a,b) \subseteq C(S).$$

Suppose $a, b \in C(S)$ and $r > 0$. Let

$$G^D(a, b; r) = G^D(N_r(a), N_r(b)).$$

We show

Theorem 3.8. *Suppose $\xi(S) \geq 1$. Let $N_S(2M, 3)$ be from Theorem A.*

1. *For any $a, b \in C(S)$ and any $c \in C(S)$ such that c lies on some $g_{a,b}$,*

$$|G^D(a, b) \cap N_\delta(c)| \leq N_S(2D, 3).$$

2. *For any $r \geq 0$, for any $a, b \in C(S)$ such that $d_S(a, b) > 2r + 2j + 1$ (where $j = 3\delta + 2$), then for any $c \in C(S)$ such that c lies on some $g_{a,b}$ and $c \notin N_{r+j}(a) \cup N_{r+j}(b)$,*

$$|G^D(a, b; r) \cap N_{2\delta}(c)| \leq N_S(2(D + M), 3)$$

for a fixed D.

3.4 Theorem A and Theorem B

Theorem 3.8 will be rephrased by Theorem B in the rest of this chapter. First we define the following.

Definition 3.9. Suppose $C \subseteq C(S)$. Let $l > 0$, $k > 1$ and $Z \subseteq S$. We say C satisfies the property $\mathcal{P}(l, k, Z)$ if there do not exist more than $k - 1$ curves in C whose projections on Z are mutually more than l apart in $C(Z)$. We say C satisfies the property $\mathcal{P}(l, k)$ if C satisfies the property $\mathcal{P}(l, k, Z)$ for all $Z \subseteq S$.

By the above definition, we rephrase the local finiteness property of the curve complex via subsurface projections by Theorem A. We prove

Theorem A. *Suppose $\xi(S) \geq 1$. Given $l > 0$ and $k > 1$, if $A \subseteq C(S)$ satisfies $\mathcal{P}(l, k)$, then there exists a computable $N_S(l, k)$ such that $|A| \leq N_S(l, k)$.*

With Theorem A when $k = 3$, we have Theorem B which follows from Corollary 3.11.

Theorem B. *Suppose $\xi(S) \geq 1$.*

1. *For any $a, b \in C(S)$ and any $c \in C(S)$ such that c lies on some $g_{a,b}$,*

$$|G^D(a,b) \cap N_\delta(c)| \leq N_S(2D, 3).$$

2. *For any $r \geq 0$, for any $a, b \in C(S)$ such that $d_S(a,b) > 2r + 2j + 1$ (where $j = 3\delta + 2$), then for any $c \in C(S)$ such that c lies on some $g_{a,b}$ and $c \notin N_{r+j}(a) \cup N_{r+j}(b)$,*

$$|G^D(a,b;r) \cap N_{2\delta}(c)| \leq N_S(2(D+M), 3).$$

3.5 Theorem A implies Theorem B

We understand the relationship between the local finiteness theorem of the curve complex and the cardinality of the slices of tight geodesics.

Definition 3.10. Suppose $x, y \in C(S)$. We let $g^w_{x,y}$ be a weak tight geodesic between x and y.

The following is a direct corollary of Lemma 3.4. (The second statement requires some typical arguments regarding Gromov-hyperbolic spaces, but those arguments are not the core of the proof.)

Corollary 3.11. *Suppose $\xi(S) \geq 1$.*

1. *For any $a, b \in C(S)$ and any $c \in C(S)$ such that c lies on some $g_{a,b}$, $G^D(a,b) \cap N_\delta(c)$ satisfies $\mathcal{P}(2D, 3)$.*

2. *For $r \geq 0$, $a, b \in C(S)$. Assume $d_S(a,b) > 2r + 2j + 1$ (where $j = 3\delta + 2$). Then for any $c \in C(S)$ such that c lies on some $g_{a,b}$ and $c \notin N_{r+j}(a) \cup N_{r+j}(b)$, $G^D(a,b;r) \cap N_{2\delta}(c)$ satisfies $\mathcal{P}(2M + 2D, 3)$.*

Proof. It suffices to show that for any x in the slices, if x projects nontrivially on some $Z \subseteq S$, $d_Z(a,x) \leq P$ or $d_Z(x,b) \leq P$ where $P = D$ for the first statement and $P = M + D$ for the second statement.

We assume $D > M$ for a more interesting discussion, the case $D = M$ is when we have tight geodesics. Since $D > M > 4\delta$ [Webb], we may exclude the case when $Z = S$.

The first statement directly follows from the definition.

For the second statement, we also recall the fact that two curves which are distance more than 2 apart fill S.

Let $x_a \in N_r(a)$, $x_b \in N_r(b)$ such that there exist $g^w_{x_a,x_b}$ which contains x. Let $g^w_{x_a,x}$ be the subsegment of $g^w_{x_a,x_b}$ between x_a and x, and g^w_{x,x_b} be the subsegment of $g^w_{x_a,x_b}$ between x and x_b.

Case 1: Suppose $d_Z(x_a, x) \le D$ and $d_Z(x, x_b) \le D$. Then since

$$\min_{a' \in N_r(a), b' \in N_r(b)} d_S(a', b') > 2,$$

by the Bounded Geodesic Image Theorem, we have

$$d_Z(a, x_a) \le M \text{ or } d_Z(x_b, b) \le M \Longrightarrow d_Z(a, x) \le D + M \text{ or } d_Z(x, b) \le D + M.$$

Case 2: Suppose not. Let us assume that $d_Z(x_a, x) \le D$ and $d_Z(x, x_b) > D$. We claim $d_Z(a, x_a) \le M$. With the claim we have

$$d_Z(a, x_a) \le d_Z(a, x_a) + d_Z(x_a, x) \le M + D.$$

Now, we prove the claim.

Since $d_Z(x, x_b) > D > M$, there exists $q \in g^w_{x,x_b}$ such that $\pi_Z(q) = \emptyset$. We show

$$q \in N_{d_S(c,b)+3\delta}(b).$$

By the hypothesis on c, we have $N_{r+2}(a) \cap N_{d_S(c,b)+3\delta}(b) = \emptyset$, so we observe that if $q \in N_{d_S(c,b)+3\delta}(b)$, then $q \notin N_{r+2}(b)$, which implies every vertex of g_{a,x_a} projects to Z nontrivially. Therefore, by the Bounded Geodesic Image Theorem, we have

$$d_Z(a, x_a) \le M.$$

$\boxed{\text{The proof of } q \in N_{d_S(c,b)+3\delta}(b)}$

We consider the 4-gon whose edges are g^w_{x,x_b}, $g_{x_b,b}$, $g_{b,c}$, and $g_{c,x}$. We consider an additional geodesic g_{c,x_b}, which decomposes the 4-gon into two triangles, and by hyperbolicity, we have

$$q \in N_{d_S(c,b)+3\delta}(b).$$

$\square$

We observe that Theorem B follows from Corollary 3.11 and Theorem A.

3.6 The proof of Theorem A

The curve graphs of $S_{1,1}$ and $S_{0,4}$ are farey graphs where vertices are identified with $\mathbb{Q} \cup \{\frac{1}{0} = \infty\}$. There is an isometry between $C(S_{1,1})$ and $C(S_{0,4})$. For a detailed treatment, see [FM12].

Suppose $x \in C(S)$. We let $C_i(x) = \{y \in C(S) | d_S(x, y) = i\}$. Now, we observe the following lemma which is the heart of the proof of Theorem A.

Lemma 3.12. *Suppose $\xi(S) \geq 1$. Let $x \in C(S)$ and $B \subseteq C_i(x)$ for $i > 1$.*

Let $Z \subset S$ such that

- *if $\xi(S) = 1$, $Z = R(x)$.*
- *if $\xi(S) > 1$, $Z \subseteq S - x$.*

Then, if B satisfies $\mathcal{P}(l, k, Z)$, there exists $B' \subseteq C_1(S)$ which satisfies $\mathcal{P}(l + 2M, k, Z)$ and with $B \subseteq \cup_{y \in B'} C_{i-1}(y)$.

Proof. The proof will be the combination of Lemma 3.4 (for $\xi(S) > 1$) and the Bounded Geodesic Image Theorem.

If $\xi(S) = 1$: Let $b \in B$. Then every vertex of $g_{x,b} - \{x\}$ projects nontrivially on $R(x)$. Therefore, if $\{b'\} = g_{x,b} \cap C_1(x)$, we have $d_{R(x)}(b', b) \leq M$ by the Bounded Geodesic Image Theorem, which implies that $B' = \cup_{w \in B} g_{x,w} \cap C_1(x)$ satisfies $\mathcal{P}(l + 2M, k, R(x))$.

By the definition of B', we observe that $B \subseteq \cup_{y \in B'} C_{i-1}(y)$.

If $\xi(S) > 1$: In this case, we make use of tight geodesics. Let $b \in B$, then by Lemma 3.4, we may take a $g^t_{x,b}$ so that if $\{b'\} = g^t_{x,b} \cap C_1(x)$ projects nontrivially on Z, every vertex of $g^t_{x,b} - \{x\}$ projects nontrivially on Z. As in the previous case, $B' = \cup_{w \in B} g^t_{x,w} \cap C_1(x)$ satisfies $\mathcal{P}(l + 2M, k, Z)$ and $B \subseteq \cup_{y \in B'} C_{i-1}(y)$. $\square$

We prove Theorem A by double induction on the complexity and the distance. First, we prove Theorem A for $\xi(S) = 1$. Indeed, we assume $S = S_{1,1}$.

Theorem 3.13. *Suppose $S = S_{1,1}$. Given $l > 0$ and $k > 1$, if $A \subseteq C(S)$ satisfies $\mathcal{P}(l, k)$, then $|A| \leq (Lk)^{l+1}$, where $L = l + 2M + 2$.*

Proof. Since A satisfies $\mathcal{P}(l,k,S)$, there are not more than $k-1$ curves in A which are mutually more than l apart in $C(S)$, so it suffices to understand a bound for $A \cap N_l(x)$ where $x \in C(S)$.

We claim that $|A \cap C_i(x)| \leq (Lk)^i$ for all $1 \leq i \leq l$ by induction on i. With the claim, we observe that

$$\begin{aligned} |A| \leq (1+\sum_{i=1}^{l}(Lk)^i)\cdot(k-1) &= \left(\sum_{i=0}^{l}(Lk)^i\right)\cdot(k-1) \\ &= \left(\frac{(Lk)^{l+1}-1}{Lk-1}\right)\cdot(k-1) \\ &\leq (Lk)^{l+1}. \end{aligned}$$

<u>Base case</u>: If $\frac{s}{t}, \frac{p}{q} \in C(S_{1,1}), i(\frac{s}{t}, \frac{p}{q}) = |sq-tp|$. We may assume $x = \frac{1}{0}$ then $C_1(x) = \mathbb{Z}$. Let T_x be the dehn twist along x. If $y \in C_1(x)$, then $d_{R(x)}(T_x^i(y), y) = |i| + 2$ by Lemma 3.2, so we have

$$|A \cap C_1(x)| \leq (l+2)(k) \leq Lk.$$

<u>Inductive step</u>: Let $B = A \cap C_i(x)$. Then B satisfies $\mathcal{P}(l,k,R(x))$. By Lemma 3.12, there exists B' which satisfies $\mathcal{P}(l+2M,k,R(x))$ and with $B \subseteq \cup_{y \in B'} C_{i-1}(y)$. By the base case, we have

$$|B'| \leq (l+2M+2)(k) \leq Lk.$$

With our inductive hypothesis, we have $|B| \leq (Lk)\cdot(Lk)^{i-1} \leq (Lk)^i$.

□

For $S_{0,4}$, analogous proof works, the only difference is that we use the half twist along x, H_x instead of T_x and use Lemma 3.2. With the same setting as in Theorem 3.13, we have $|A \cap C_i(x)| \leq (2(L-1)k)^i \leq (2Lk)^i$ for all $1 \leq i \leq l$. Therefore,

$$N_{S_{0,4}}(l,k) = (2Lk)^{l+1}.$$

Now, we complete the proof of Theorem A.

Proof of Theorem A. Let $N_{S'}(l,k) = \max_{\xi(S_{g,n})<\xi(S)} N_{S_{g,n}}(l,k)$.

Since A satisfies $\mathcal{P}(l,k,S)$, A is contained in $k-1$ balls of radius l in $C(S)$. Therefore, it suffices to understand a bound for $A \cap N_l(x)$ where $x \in C(S)$.

Let $L = l + 2M$. We claim that $|A \cap C_i(x)| \leq (2N_{S'}(L,k))^i$ for all $1 \leq i \leq l$ by induction on i. With the claim, we observe that

$$\begin{aligned} |A| &\leq \left(1 + \sum_{i=1}^{l} (2N_{S'}(L,k))^i\right) \cdot (k-1) \\ &= \left(\sum_{i=0}^{l} (2N_{S'}(L,k))^i\right) \cdot (k-1) \\ &= \left(\frac{(2N_{S'}(L,k))^{l+1} - 1}{2N_{S'}(L,k) - 1}\right) \cdot (k-1). \end{aligned}$$

Since $k - 1 \leq 2N_{S'}(L,k) - 1$, we have

$$|A| \leq (2N_{S'}(L,k))^{l+1}.$$

Base case: Suppose $S - x = \{S_1, S_2\}$. Then we may assume $\xi(S_1), \xi(S_2) \geq 1$ since $C(S_{0,3}) = \emptyset$. (If $S - x$ has one component, we let $S - x = \{S_1\}$, and treat $S_2 = \emptyset$.) Therefore,

$$|A \cap C_1(x)| \leq N_{S_1}(l,k) + N_{S_2}(l,k) \leq 2N_{S'}(l,k) \leq 2N_{S'}(L,k).$$

Inductive step: Let $B = A \cap C_i(x)$. Since B satisfies $\mathcal{P}(l+2M,k,Z)$ for all $Z \subseteq S - x$, by Lemma 3.12, there exists $B' \subseteq C_1(x)$ which satisfies $\mathcal{P}(l+2M,k,Z)$ for all $Z \subseteq S - x$ and with $B \subseteq \cup_{y \in B'} C_{i-1}(y)$. Therefore,

$$|B'| \leq 2N_{S'}(l+2M,k) = 2N_{S'}(L,k)$$

and we have

$$|B| \leq 2N_{S'}(L,k) \cdot (2N_{S'}(L,k))^{i-1} = (2N_{S'}(L,k))^i.$$

Therefore, we have

$$|A| \leq (2N_{S'}(L,k))^{l+1}.$$

and we are done. $\square$

REFERENCES

[Aou13] Tarik Aougab. Uniform hyperbolicity of the graphs of curves. Geom. Topol., 17(5):2855–2875, 2013.

[BCM12] Jeffrey F. Brock, Richard D. Canary, and Yair N. Minsky. The classification of Kleinian surface groups, II: The ending lamination conjecture. Ann. of Math. (2), 176(1):1–149, 2012.

[BF08] Gregory C. Bell and Koji Fujiwara. The asymptotic dimension of a curve graph is finite. J. Lond. Math. Soc. (2), 77(1):33–50, 2008.

[Bow06] Brian H. Bowditch. Intersection numbers and the hyperbolicity of the curve complex. J. Reine Angew. Math., 598:105–129, 2006.

[Bow07] Brian H. Bowditch. Length bounds on curves arising from tight geodesics. Geom. Funct. Anal., 17(4):1001–1042, 2007.

[Bow08] Brian H. Bowditch. Tight geodesics in the curve complex. Invent. Math., 171(2):281–300, 2008.

[Bow14] Brian Bowditch. Uniform hyperbolicity of the curve graphs. Pacific J. Math., 269(2):269–280, 2014.

[FM12] Benson Farb and Dan Margalit. A primer on mapping class groups, volume 49 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 2012.

[Har81] W. J. Harvey. Boundary structure of the modular group. In Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), volume 97 of Ann. of Math. Stud., pages 245–251. Princeton Univ. Press, Princeton, N.J., 1981.

[HPW] Sebastian Hensel, Piotr Przytycki, and Richard C. H. Webb. Slim unicorns and uniform hyperbolicity for arc graphs and curve graphs. arXiv:1301.5577.

[Lea02] Jason Paige Leasure. Geodesics in the complex of curves of a surface. ProQuest LLC, Ann Arbor, MI, 2002. Thesis (Ph.D.)–The University of Texas at Austin.

[Luo10] Feng Luo. Simple loops on surfaces and their intersection numbers. J. Differential Geom., 85(1):73–115, 2010.

[MCS] Kasra Rafi Matt Clay and Saul Schlelmer. Uniform hyperbolicity of the curve graph via surgery sequences. arXiv:1302.5519.

[Min10] Yair Minsky. The classification of Kleinian surface groups. I. Models and bounds. Ann. of Math. (2), 171(1):1–107, 2010.

[MM99] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. Invent. Math., 138(1):103–149, 1999.

[MM00] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves. II. Hierarchical structure. Geom. Funct. Anal., 10(4):902–974, 2000.

[Sha12] Kenneth J. Shackleton. Tightness and computing distances in the curve complex. Geom. Dedicata, 160:243–259, 2012.

[Wata] Yohsuke Watanabe. Intersection numbers in the curve complex via subsurface projections. arXiv:1407.3321.

[Watb] Yohsuke Watanabe. A polygon decomposition of a surface by filling curves and tight geodesics. http://www.math.utah.edu/~ywatanab/.

[Weba] Richard C. H. Webb. Combinatorics of tight geodesics and stable lengths. arXiv:1305.3566.

[Webb] Richard C. H. Webb. A short proof of the bounded geodesic image theorem. arXiv:1301.6187.